Double bremsstrahlung from high-energy electron in the atomic field.

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Abstract

The differential cross section of double bremsstrahlung from high-energy electron in the electric field of heavy atoms is derived. The results are obtained with the exact account of the atomic field by means of the quasiclassical approximation to the wave functions and the Green's function in the external field. It is shown that the Coulomb corrections to the differential cross section (the difference between the exact result and the result obtained in the leading Born approximation) correspond to small momentum transfers. The Coulomb corrections to the differential cross section of double bremsstrahlung are accumulated in the factor, which coincides with the corresponding factor in the differential cross section of single bremsstrahlung. At small momentum transfer, this factor is very sensitive to the parameters of screening while the Coulomb corrections to the spectrum have the universal form.

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I. INTRODUCTION

To search for New Physics in precision experiments, it is necessary to know with high accuracy the cross sections of the main background processes. In particular, it is necessary to know the cross sections of single high-energy bremsstrahlung and particle-antiparticle photoproduction in the electric field of a heavy nucleus or atom. These processes play a dominant role when considering electromagnetic showers in detectors. In many cases they also give significant part of the radiative corrections. In the Born approximation, the cross sections of both processes are known for arbitrary energies of particles [1, 2] (see also Ref. [3]). However, for large Z the Coulomb corrections (i.e. the contribution of higher-order terms in the parameter $\eta = Z\alpha$) to the cross section are very important (here Z is the atomic charge number, $\alpha = e^2 \approx 1/137$ is the fine-structure constant, e is the electron charge, $\hbar = c = 1$). Though there are formal expressions for the cross sections exact in η and energies of particles [4], their use for numerical computations becomes very difficult at high energies [5].

Fortunately, at high energies of initial particles, the main contribution to the cross section comes from small angles of the final particle momenta with respect to the incident direction. In this case typical angular momenta are large ($l \sim E/\Delta \gg 1$, where E is energy and Δ is the momentum transfer). Therefore, the quasiclassical approximation, which accounts for large angular momenta contributions, becomes applicable. Using the quasiclassical wave functions and the quasiclassical Green's functions of the Dirac equation in the external field, one can drastically simplify calculations. The celebrated Furry-Sommerfeld-Maue wave functions [6, 7] (see also Ref. [3]) is nothing else but the leading-order quasiclassical wave functions for the Coulomb field. The quasiclassical Green's function have been derived in Ref. [8] for the case of a pure Coulomb field, in Ref. [9] for an arbitrary spherically symmetric field, in Ref. [10] for any localized field, and in Ref. [11] for combined strong laser and atomic fields.

For pair photoproduction and single bremsstrahlung, the cross sections in the leading quasiclassical approximation have been obtained in Refs. [12–16]. The first quasiclassical corrections to the spectra of both processes have been obtained in Refs. [17–20]. Recently, the first quasiclassical corrections to the fully differential cross sections were obtained in Ref. [21] for e^+e^- pair photoproduction, in Ref. [22] for $\mu^+\mu^-$ pair photoproduction, and in Ref. [23] for single bremsstrahlung from high-energy electrons and muons in an atomic field. The account for the first quasiclassical corrections allows one to determine quantitatively the

charge asymmetry in these processes (the asymmetry of the cross sections with respect to the permutation of particle and antiparticle). This asymmetry is absent in the cross section calculated in the leading quasiclassical approximation.

Influence of screening (the difference between the atomic field and the Coulomb field of a nucleus) on the Coulomb corrections to e^+e^- pair photoproduction cross section is small for the differential cross section and for the total cross section as well [13], see Ref. [17], where the effect of screening has been investigated quantitatively. However, screening is important for the Born term. A role of screening in single bremsstrahlung in the atomic field is different. It is shown in Refs. [14, 18] that the Coulomb corrections to the differential cross section are very susceptible to screening. However, the Coulomb corrections to the cross section integrated over the momentum of final charged particle (electron or muon) are independent of screening in the leading approximation over a small parameter $1/mr_{scr}$ [18], where $r_{scr} \sim Z^{-1/3}(m\alpha)^{-1}$ is a screening radius and m is the electron mass.

Investigation of high-energy e^+e^- photoproduction accompanied by bremsstrahlung and double bremsstrahlung from electrons in the electric field of a heavy atom (i.e., the processes $\gamma_1 Z \to e^+e^-\gamma_2 Z$ and $e^\pm Z \to \gamma_1\gamma_2 e^\pm Z$, respectively) is even more complicated task. The process $\gamma_1 Z \to e^+e^-\gamma_2 Z$ is a significant part of the radiative corrections to e^+e^- photoproduction as well as a noticeable background to such processes as Delbrück scattering [24]. This process should be taken into account at the consideration of the electromagnetic showers in the matter. During a long time only a few papers, related to this process, have been published [25, 26]. In those papers the Born approximation was used. Very recently, using the quasiclassical approximation, the cross section of the process $\gamma_1 Z \to e^+e^-\gamma_2 Z$ at high energies was derived exactly in the parameter η [27]. It was shown that, apart from the region of very small momentum transfer, account of the Coulomb corrections for heavy atoms drastically change the result.

As to the double bremsstrahlung cross section from electron in an atomic field, it has been investigated either at low electron energies [28, 29] or for any electron energies but in the Born approximation [30]. In the present paper we use the quasiclassical approximation to derive the exact in η differential cross section of double bremsstrahlung from high energy electron in an atomic field. We take into account the effect of screening and show that the Coulomb corrections to the cross section are, in general, very sensitive to this effect. Moreover, the Coulomb corrections to the double bremsstrahlung cross section are accumulated in the

factor which coincides with the corresponding factor in the differential cross section of single bremsstrahlung. This allows us to formulate a recipe for the calculation of the multiple bremsstrahlung amplitudes.

II. GENERAL DISCUSSION

The differential cross section of double bremsstrahlung in the electric field of a heavy atom reads [3]

$$d\sigma = \frac{\alpha^2}{(2\pi)^6}\omega_1\omega_2 q\varepsilon_q d\omega_1 d\omega_2\, d\Omega_{\boldsymbol{k}_1}\, d\Omega_{\boldsymbol{k}_2} d\Omega_{\boldsymbol{q}}\, |M|^2\,, \tag{1}$$

where $d\Omega_{\boldsymbol{k}_1}$, $d\Omega_{\boldsymbol{k}_2}$, and $d\Omega_{\boldsymbol{q}}$ are the solid angles corresponding to the photon momentum $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, and the final charged particle momentum $\boldsymbol{q}$, $\varepsilon_q = \varepsilon_p - \omega_1 - \omega_2$ is the final charge particle energy, $\varepsilon_p = \sqrt{\boldsymbol{p}^2 + m^2}$, $\varepsilon_q = \sqrt{\boldsymbol{q}^2 + m^2}$. Below we assume that $\varepsilon_p \gg m$ and $\varepsilon_q \gg m$. The matrix element M reads

$$\begin{aligned}
M &= M^{(1)} + M^{(2)}\,,\\
M^{(1)} &= -\iint d\boldsymbol{r}_1 d\boldsymbol{r}_2\, e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}_1 - i\boldsymbol{k}_2\cdot\boldsymbol{r}_2}\, \bar{u}^{(-)}_{\boldsymbol{q}}(\boldsymbol{r}_2)\hat{e}_2^* G(\boldsymbol{r}_2, \boldsymbol{r}_1|\varepsilon_p - \omega_1)\hat{e}_1^*\, u^{(+)}_{\boldsymbol{p}}(\boldsymbol{r}_1)\,,\\
M^{(2)} &= M^{(1)}(\boldsymbol{k}_1 \leftrightarrow \boldsymbol{k}_2\,,\, \omega_1 \leftrightarrow \omega_2\,,\, \boldsymbol{e}_1 \leftrightarrow \boldsymbol{e}_2)\,,
\end{aligned} \tag{2}$$

where $\hat{e} = \gamma^\nu e_\nu = -\boldsymbol{\gamma}\cdot\boldsymbol{e}$, γ^ν are the Dirac matrices, $u^{(+)}_{\boldsymbol{p}}(\boldsymbol{r})$ and $u^{(-)}_{\boldsymbol{q}}(\boldsymbol{r})$ are the solutions of the Dirac equation in the atomic potential $V(r)$, $\boldsymbol{e}_{1\,2}$ are the photon polarization vectors, and $G(\boldsymbol{r}_2, \boldsymbol{r}_1|\varepsilon)$ is the Green's function of the Dirac equation in the potential $V(r)$. The superscripts $(-)$ and $(+)$ remind us that the asymptotic forms of $u^{(-)}_{\boldsymbol{q}}(\boldsymbol{r})$ and $u^{(+)}_{\boldsymbol{p}}(\boldsymbol{r})$ at large $\boldsymbol{r}$ contain, in addition to the plane wave, the spherical convergent and divergent waves, respectively. It is convenient to write the contribution $M^{(1)}$ in Eq. (2) in terms of the Green's function $D(\boldsymbol{r}_2,\, \boldsymbol{r}_1|\varepsilon)$ of the "squared" Dirac equation,

$$G(\boldsymbol{r}_2,\, \boldsymbol{r}_1|\varepsilon) = (\hat{\mathcal{P}} + m)D(\boldsymbol{r}_2,\, \boldsymbol{r}_1|\varepsilon), \quad D(\boldsymbol{r}_2,\, \boldsymbol{r}_1|\varepsilon) = \langle \boldsymbol{r}_2|\frac{1}{\hat{\mathcal{P}}^2 - m^2 + i0}|\boldsymbol{r}_1\rangle\,, \tag{3}$$

where $\hat{\mathcal{P}} = \gamma^\nu \mathcal{P}_\nu$, $\mathcal{P}_\nu = \big(\varepsilon - V(r), i\boldsymbol{\nabla}\big)$. Substituting Eq. (3) in Eq. (2), performing integration by parts and using the Dirac equation, we obtain

$$\begin{aligned}
M^{(1)} = &-\iint d\boldsymbol{r}_1 d\boldsymbol{r}_2\, e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}_1 - i\boldsymbol{k}_2\cdot\boldsymbol{r}_2}\, \bar{u}^{(-)}_{\boldsymbol{q}}(\boldsymbol{r}_2)\hat{e}_2^* D(\boldsymbol{r}_2, \boldsymbol{r}_1|\varepsilon_p - \omega_1)\\
&\times [2i\boldsymbol{e}_1^*\cdot\boldsymbol{\nabla} + \hat{e}_1^*\hat{k}_1]\, u^{(+)}_{\boldsymbol{p}}(\boldsymbol{r}_1)\,.
\end{aligned} \tag{4}$$

The Green's function $D(\boldsymbol{r}_2,\, \boldsymbol{r}_1|\varepsilon)$ and the wave functions $u_{\boldsymbol{p}}^{(+)}(\boldsymbol{r})$ and $u_{\boldsymbol{q}}^{(-)}(\boldsymbol{r})$ have the form [22, 23]

$$
\begin{aligned}
& D(\boldsymbol{r}_2,\, \boldsymbol{r}_1|\varepsilon) = d_0(\boldsymbol{r}_2, \boldsymbol{r}_1) + \boldsymbol{\alpha}\cdot \boldsymbol{d}_1(\boldsymbol{r}_2, \boldsymbol{r}_1) + \boldsymbol{\Sigma}\cdot \boldsymbol{d}_2(\boldsymbol{r}_2, \boldsymbol{r}_1)\,, \\
& \bar{u}_{\boldsymbol{q}}^{(-)}(\boldsymbol{r}) = \bar{u}_{\boldsymbol{q}}[f_0(\boldsymbol{r}, \boldsymbol{q}) - \boldsymbol{\alpha}\cdot \boldsymbol{f}_1(\boldsymbol{r}, \boldsymbol{q}) - \boldsymbol{\Sigma}\cdot \boldsymbol{f}_2(\boldsymbol{r}, \boldsymbol{q})]\,, \\
& u_{\boldsymbol{p}}^{(+)}(\boldsymbol{r}) = [g_0(\boldsymbol{r}, \boldsymbol{p}) - \boldsymbol{\alpha}\cdot \boldsymbol{g}_1(\boldsymbol{r}, \boldsymbol{p}) - \boldsymbol{\Sigma}\cdot \boldsymbol{g}_2(\boldsymbol{r}, \boldsymbol{p})]u_{\boldsymbol{p}}\,, \\
& u_{\boldsymbol{p}} = \sqrt{\frac{\varepsilon_p + m}{2\varepsilon_p}} \begin{pmatrix} \phi \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\varepsilon_p + m}\phi \end{pmatrix}\,, \quad u_{\boldsymbol{q}} = \sqrt{\frac{\varepsilon_q + m}{2\varepsilon_q}} \begin{pmatrix} \chi \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{q}}{\varepsilon_q + m}\chi \end{pmatrix}\,,
\end{aligned} \tag{5}
$$

where ϕ and χ are spinors, $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, $\boldsymbol{\Sigma} = \gamma^0\gamma^5\boldsymbol{\gamma}$, $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, and $\boldsymbol{\sigma}$ are the Pauli matrices. The coefficients d_0, $\boldsymbol{d}_1$, f_0, $\boldsymbol{f}_1$, g_0 and $\boldsymbol{g}_1$ in the leading quasiclassical approximation, as well as the first quasiclassical corrections to d_0, f_0 and g_0, were derived in Ref. [10] for arbitrary atomic potential $V(r)$. The first quasiclassical corrections to $\boldsymbol{d}_1$, $\boldsymbol{f}_1$ and $\boldsymbol{g}_1$, together with the leading quasiclassical terms of $\boldsymbol{d}_2$, $\boldsymbol{f}_2$ and $\boldsymbol{g}_2$, were derived in Ref. [23]. We perform calculation of the double bremsstrahlung cross section in the leading quasiclassical approximation. In this case it is sufficient to take into account the terms d_0, $\boldsymbol{d}_1$, f_0, $\boldsymbol{f}_1$, g_0 and $\boldsymbol{g}_1$ in the leading quasiclassical approximation and neglect the contributions of $\boldsymbol{d}_2$, $\boldsymbol{f}_2$ and $\boldsymbol{g}_2$ [21–23]. Within this accuracy we have for d_0 and $\boldsymbol{d}_1$

$$
\begin{aligned}
& d_0(\boldsymbol{r}_2, \boldsymbol{r}_1) = \frac{ie^{i\kappa r}}{4\pi^2 r}\int d\boldsymbol{Q}\exp\left[iQ^2 - ir\int_0^1 dx V(\boldsymbol{R})\right]\,, \\
& \boldsymbol{d}_1(\boldsymbol{r}_2, \boldsymbol{r}_1) = -\frac{i}{2\varepsilon}(\boldsymbol{\nabla}_1 + \boldsymbol{\nabla}_2)d_0(\boldsymbol{r}_2, \boldsymbol{r}_1)\,, \\
& \boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1\,, \quad \boldsymbol{R} = \boldsymbol{r}_1 + x\boldsymbol{r} + \boldsymbol{Q}\sqrt{\frac{2x(1-x)r}{\kappa}}\,, \quad \kappa = \sqrt{\varepsilon^2 - m^2}\,,
\end{aligned} \tag{6}
$$

where $\boldsymbol{Q}$ is a two-dimensional vector perpendicular to the vector $\boldsymbol{r}_2 - \boldsymbol{r}_1$. The terms f_0 and $\boldsymbol{f}_1$ are

$$
\begin{aligned}
& f_0(\boldsymbol{r}, \boldsymbol{q}) = -\frac{i}{\pi}e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\int d\boldsymbol{Q}\exp\left[iQ^2 - i\int_0^\infty dx V(\boldsymbol{r}_q)\right]\,, \\
& \boldsymbol{f}_1(\boldsymbol{r}, \boldsymbol{q}) = \frac{1}{2\varepsilon_q}(i\boldsymbol{\nabla} - \boldsymbol{q})f_0(\boldsymbol{r}, \boldsymbol{q})\,, \\
& \boldsymbol{r}_q = \boldsymbol{r} + x\boldsymbol{n}_q + \boldsymbol{Q}\sqrt{\frac{2x}{\varepsilon_q}}\,, \quad \boldsymbol{Q}\cdot\boldsymbol{n}_q = 0\,, \quad \boldsymbol{n}_q = \boldsymbol{q}/q\,.
\end{aligned} \tag{7}
$$

The expressions for g_0 and $\boldsymbol{g}_1$ follow from the relations

$$
g_0(\boldsymbol{r}, \boldsymbol{p}) = f_0(\boldsymbol{r}, -\boldsymbol{p})\,, \quad \boldsymbol{g}_1(\boldsymbol{r}, \boldsymbol{p}) = \boldsymbol{f}_1(\boldsymbol{r}, -\boldsymbol{p})\,. \tag{8}
$$

It is convenient to calculate the matrix element for definite helicities of the particles. Let μ_p, μ_q, λ_1, and λ_2 be the signs of the helicities of initial electron, final electron, and radiated photons, respectively. We fix the coordinate system so that $\boldsymbol{\nu} \equiv \boldsymbol{n}_p = \boldsymbol{p}/p$ is directed along z-axis and $\boldsymbol{q}$ lies in the xz plane with $q_x > 0$. Denoting helicities by the subscripts, we have

$$
\begin{aligned}
&\phi = \frac{1+\mu_p \boldsymbol{\sigma}\cdot\boldsymbol{n}_p}{4}\begin{pmatrix}1+\mu_p\\ 1-\mu_p\end{pmatrix},\\
&\chi = \frac{1+\mu_q \boldsymbol{\sigma}\cdot\boldsymbol{n}_q}{4\cos(\theta_q/2)}\begin{pmatrix}1+\mu_q\\ 1-\mu_q\end{pmatrix} \approx \frac{1}{4}\left(1+\frac{\theta_q^2}{8}\right)(1+\mu_q \boldsymbol{\sigma}\cdot\boldsymbol{n}_q)\begin{pmatrix}1+\mu_q\\ 1-\mu_q\end{pmatrix},\\
&\boldsymbol{e}_1 = \boldsymbol{s}_{\lambda_1} - (\boldsymbol{s}_{\lambda_1}\cdot\boldsymbol{\theta}_{k_1})\boldsymbol{\nu}\,, \quad \boldsymbol{e}_2 = \boldsymbol{s}_{\lambda_2} - (\boldsymbol{s}_{\lambda_2}\cdot\boldsymbol{\theta}_{k_2})\boldsymbol{\nu}\,,\\
&\boldsymbol{s}_\lambda = \frac{1}{\sqrt{2}}(\boldsymbol{e}_x + i\lambda\boldsymbol{e}_y)\,,
\end{aligned}
\tag{9}
$$

where $\boldsymbol{\theta}_q = \boldsymbol{q}_\perp/q$, $\boldsymbol{\theta}_{k_1} = \boldsymbol{k}_{1\perp}/\omega_1$, and $\boldsymbol{\theta}_{k_2} = \boldsymbol{k}_{2\perp}/\omega_2$, the notation $\boldsymbol{X}_\perp = \boldsymbol{X} - (\boldsymbol{\nu}\cdot\boldsymbol{X})\boldsymbol{\nu}$ for any vector $\boldsymbol{X}$ is used. Below we assume that $\theta_q \ll 1$, $\theta_{k_1} \ll 1$, and $\theta_{k_2} \ll 1$. The unit vectors $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are directed along $\boldsymbol{q}_\perp$ and $\boldsymbol{p}\times\boldsymbol{q}$. In the expressions for $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in (9), the terms of the order $O(\theta_{k_1}^2)$ and $O(\theta_{k_2}^2)$ are omitted. For the matrix $\mathcal{F} = u_{\boldsymbol{p}\,\mu_p}\bar{u}_{\boldsymbol{q}\,\mu_q}$ we have [23]

$$
\begin{aligned}
&\mathcal{F} = \frac{1}{8}(a_{\mu_p\mu_q} + \boldsymbol{\Sigma}\cdot\boldsymbol{b}_{\mu_p\mu_q})[\gamma^0(1+PQ) + \gamma^0\gamma^5(P+Q) + (1-PQ) - \gamma^5(P-Q)],\\
&P = \frac{\mu_p p}{\varepsilon_p + m}\,, \quad Q = \frac{\mu_q q}{\varepsilon_q + m}\,.
\end{aligned}
\tag{10}
$$

Here $a_{\mu_p\mu_q}$ and $\boldsymbol{b}_{\mu_p\mu_q}$ are

$$
\begin{aligned}
&a_{\mu\mu} = 1 - \frac{\theta_q^2}{8}\,, \quad a_{\mu\bar{\mu}} = -\frac{\mu}{\sqrt{2}}\boldsymbol{s}_\mu\cdot\boldsymbol{\theta}_q\,,\\
&\boldsymbol{b}_{\mu\mu} = \mu\left(1-\frac{\theta_q^2}{8}\right)\boldsymbol{\nu} + \frac{\mu}{2}\boldsymbol{\theta}_q - \frac{i}{2}[\boldsymbol{\theta}_q\times\boldsymbol{\nu}]\,,\\
&\boldsymbol{b}_{\mu\bar{\mu}} = \sqrt{2}\boldsymbol{s}_\mu - \frac{1}{\sqrt{2}}(\boldsymbol{s}_\mu\cdot\boldsymbol{\theta}_q)\boldsymbol{\nu}\,, \quad \boldsymbol{s}_\mu = \frac{1}{\sqrt{2}}(\boldsymbol{e}_x + i\mu\boldsymbol{e}_y)\,,
\end{aligned}
\tag{11}
$$

where $\bar{\mu} = -\mu$. The matrix element M, Eq. (4), can be written as follows

$$
\begin{aligned}
M^{(1)} = &-\iint d\boldsymbol{r}_1 d\boldsymbol{r}_2\, e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}_1 - i\boldsymbol{k}_2\cdot\boldsymbol{r}_2}\,\mathrm{Tr}\,[f_0\hat{e}_2^* d_0\Theta\, g_0 - \boldsymbol{\alpha}\cdot\boldsymbol{f}_1\hat{e}_2^* d_0\Theta\, g_0\\
&+ f_0\hat{e}_2^*\boldsymbol{\alpha}\cdot\boldsymbol{d}_1\Theta\, g_0 - f_0\hat{e}_2^* d_0\Theta\,\boldsymbol{\alpha}\cdot\boldsymbol{g}_1]\mathcal{F}\,,\\
&\Theta = 2i\boldsymbol{e}_1^*\cdot\boldsymbol{\nabla} + \hat{e}_1^*\hat{k}_1\,.
\end{aligned}
\tag{12}
$$

Here the functions d_0 and $\boldsymbol{d}_1$ are calculated at $\varepsilon = \varepsilon_p - \omega_1$. Note that only the terms with $(P+Q)$ and $(1+PQ)$ in $\mathcal{F}$, Eq. (10), contribute to the matrix element (12) due to the trace over γ-matrices. Below we calculate the matrix element M for the atomic potential $V(r)$, which includes the effect of screening.

III. MATRIX ELEMENT AND CROSS SECTION

The calculation of the matrix element (2) is performed in the same way as in Ref.[18]. Some details of this calculation are given in Appendix. The final result is:

$$
\begin{aligned}
&M_{\mu_p\mu_q\lambda_1\lambda_2} = -\boldsymbol{A}(\boldsymbol{\Delta})\cdot\Big[\boldsymbol{T}_{\mu_p\mu_q\lambda_1\lambda_2}(\boldsymbol{k}_1,\boldsymbol{k}_2) + \boldsymbol{T}_{\mu_p\mu_q\lambda_2\lambda_1}(\boldsymbol{k}_2,\boldsymbol{k}_1)\Big]\,,\\
&\boldsymbol{A}(\boldsymbol{\Delta}) = -i\int d\boldsymbol{r}\exp\left[-i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\chi(\rho)\right]\boldsymbol{\nabla}_\perp V(r)\,,\quad \chi(\rho) = \int_{-\infty}^{\infty} V(\sqrt{z^2+\rho^2})dz\,,\\
&\boldsymbol{T}_{++++}(\boldsymbol{k}_1,\boldsymbol{k}_2) = p\left[(\boldsymbol{e}^*\cdot\boldsymbol{\theta}_{k_1})(\boldsymbol{e}^*\cdot\boldsymbol{\theta}_{k_2q})\boldsymbol{j}_0 + N_1(\boldsymbol{e}^*\cdot\boldsymbol{\theta}_{k_1})\boldsymbol{e}^* + N_3(\boldsymbol{e}^*\cdot\boldsymbol{\theta}_{k_2q})\boldsymbol{e}^*\right]\,,\\
&\boldsymbol{T}_{+++-}(\boldsymbol{k}_1,\boldsymbol{k}_2) = \left[p(\boldsymbol{e}^*\cdot\boldsymbol{\theta}_{k_1})(\boldsymbol{e}\cdot\boldsymbol{\theta}_{k_2}) - \frac{m^2\omega_1}{2pq}\right]\boldsymbol{j}_0 + p(N_2+N_3)(\boldsymbol{e}^*\cdot\boldsymbol{\theta}_{k_1})\boldsymbol{e}\\
&\quad + N_3(\boldsymbol{e},\, p\boldsymbol{\theta}_{k_2} - \boldsymbol{\Delta}_\perp)\boldsymbol{e}^*\,,\\
&\boldsymbol{T}_{++-+}(\boldsymbol{k}_1,\boldsymbol{k}_2) = \varkappa\left[(\boldsymbol{e}\cdot\boldsymbol{\theta}_{k_1})(\boldsymbol{e}^*\cdot\boldsymbol{\theta}_{k_2q})\boldsymbol{j}_0 + N_1(\boldsymbol{e}\cdot\boldsymbol{\theta}_{k_1})\boldsymbol{e}^* + N_3(\boldsymbol{e}^*\cdot\boldsymbol{\theta}_{k_2q})\boldsymbol{e}\right]\,,\\
&\boldsymbol{T}_{++--}(\boldsymbol{k}_1,\boldsymbol{k}_2) = q\left[(\boldsymbol{e}\cdot\boldsymbol{\theta}_{k_1})(\boldsymbol{e}\cdot\boldsymbol{\theta}_{k_2q})\boldsymbol{j}_0 + N_1(\boldsymbol{e}\cdot\boldsymbol{\theta}_{k_1})\boldsymbol{e} + N_3(\boldsymbol{e}\cdot\boldsymbol{\theta}_{k_2q})\boldsymbol{e}\right]\,,\\
&\boldsymbol{T}_{+-++}(\boldsymbol{k}_1,\boldsymbol{k}_2) = -\frac{m(\omega_1+\omega_2)}{\sqrt{2}q}\left[(\boldsymbol{e}^*\cdot\boldsymbol{\theta}_{k_1})\boldsymbol{j}_0 + N_3\boldsymbol{e}^*\right] - \frac{m\omega_1}{\sqrt{2}p}\left[(\boldsymbol{e}^*\cdot\boldsymbol{\theta}_{k_2k_1})\boldsymbol{j}_0 + N_2\boldsymbol{e}^*\right]\,,\\
&\boldsymbol{T}_{+-+-}(\boldsymbol{k}_1,\boldsymbol{k}_2) = -\frac{m\omega_1}{\sqrt{2}p}\left[(\boldsymbol{e}\cdot\boldsymbol{\theta}_{k_2q})\boldsymbol{j}_0 + N_1\boldsymbol{e}\right]\,,\\
&\boldsymbol{T}_{+--+}(\boldsymbol{k}_1,\boldsymbol{k}_2) = -\frac{m\omega_2}{\sqrt{2}q}\left[(\boldsymbol{e}\cdot\boldsymbol{\theta}_{k_1})\boldsymbol{j}_0 + N_3\boldsymbol{e}\right]\,,\\
&\boldsymbol{T}_{+---}(\boldsymbol{k}_1,\boldsymbol{k}_2) = 0\,,\\
&\boldsymbol{T}_{\bar\mu_p\bar\mu_q\bar\lambda_1\bar\lambda_2}(\boldsymbol{k}_1,\boldsymbol{k}_2) = \bar\mu_p\bar\mu_q\boldsymbol{T}_{\mu_p\mu_q\lambda_1\lambda_2}(\boldsymbol{k}_1,\boldsymbol{k}_2)\,|_{\boldsymbol{e}\leftrightarrow\boldsymbol{e}^*}\,,\quad \bar\mu = -\mu\,,\quad \bar\lambda = -\lambda\,.
\end{aligned}
\tag{13}
$$

Here we use the following notation

$$
\begin{aligned}
&\boldsymbol{e} = \frac{1}{\sqrt{2}}(\boldsymbol{e}_x + i\boldsymbol{e}_y)\,,\quad \boldsymbol{\theta}_{k_2q} = \boldsymbol{\theta}_{k_2} - \boldsymbol{\theta}_q\,,\quad \boldsymbol{\theta}_{k_2k_1} = \boldsymbol{\theta}_{k_2} - \boldsymbol{\theta}_{k_1}\,,\\
&\varkappa = p - \omega_1\,,\quad \boldsymbol{\Delta} = \boldsymbol{q} + \boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{p}\,,\quad \boldsymbol{\Delta}_\perp = q\boldsymbol{\theta}_q + \omega_1\boldsymbol{\theta}_{k_1} + \omega_2\boldsymbol{\theta}_{k_2}\,,\\
&\boldsymbol{j}_0 = \frac{4}{a_1a_2a_3a_4}\{a_3[(p+q)\boldsymbol{\Delta}_\perp - 2pq\boldsymbol{\theta}_q] + a_1\omega_2(\boldsymbol{\Delta}_\perp + 2q\boldsymbol{\theta}_{k_2q})\}\,,\\
&N_1 = \frac{4}{a_2a_3}\,,\quad N_2 = \frac{4}{a_3a_4}\,,\quad N_3 = \frac{4}{a_1a_4}\,,
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= -\frac{\omega_1}{\varkappa}[(\boldsymbol{\Delta}_\perp - p\boldsymbol{\theta}_{k_1})^2 + m^2] - \frac{p\omega_2}{q\varkappa}[q^2\boldsymbol{\theta}_{k_2q}^2 + m^2]\,,\\
a_2 &= \frac{\omega_2}{\varkappa}[(\boldsymbol{\Delta}_\perp + q\boldsymbol{\theta}_{k_2q})^2 + m^2] + \frac{q\omega_1}{p\varkappa}[p^2\boldsymbol{\theta}_{k_1}^2 + m^2]\,,\\
a_3 &= \frac{\omega_1}{p}[p^2\boldsymbol{\theta}_{k_1}^2 + m^2]\,, \quad a_4 = \frac{\omega_2}{q}[q^2\boldsymbol{\theta}_{k_2q}^2 + m^2]\,.
\end{aligned}
\tag{14}
$$

Within our accuracy, one can replace p and q in Eqs. (13) and (14) by ε_p and ε_q. The vector $\boldsymbol{A}(\boldsymbol{\Delta})$ is obviously parallel to the vector $\boldsymbol{\Delta}_\perp$,

$$
\boldsymbol{A}(\boldsymbol{\Delta}) = A_0(\boldsymbol{\Delta})\,\boldsymbol{\Delta}_\perp\,, \quad A_0(\boldsymbol{\Delta}) = -\frac{i}{\Delta_\perp^2}\int d\boldsymbol{r}\exp\left[-i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\chi(\rho)\right]\boldsymbol{\Delta}_\perp\cdot\boldsymbol{\nabla}_\perp V(r)\,, \tag{15}
$$

so that we can write the amplitude $M_{\mu_p\mu_q\lambda_1\lambda_2}$ as

$$
\begin{aligned}
&M_{\mu_p\mu_q\lambda_1\lambda_2} = -A_0(\boldsymbol{\Delta})\,\mathcal{T}_{\mu_p\mu_q\lambda_1\lambda_2}\,,\\
&\mathcal{T}_{\mu_p\mu_q\lambda_1\lambda_2} = \boldsymbol{\Delta}_\perp\cdot\Big[\boldsymbol{T}_{\mu_p\mu_q\lambda_1\lambda_2}(\boldsymbol{k}_1,\boldsymbol{k}_2) + \boldsymbol{T}_{\mu_p\mu_q\lambda_2\lambda_1}(\boldsymbol{k}_2,\boldsymbol{k}_1)\Big]\,.
\end{aligned}
\tag{16}
$$

The amplitude $M_{\mu_p\mu_q\lambda_1\lambda_2}$ is exact in the potential $V(r)$. Whole dependence of this amplitude on the potential $V(r)$ is contained in the factor $A_0(\boldsymbol{\Delta})$. In the Born approximation we have

$$
A_0^B(\boldsymbol{\Delta}) = V_F(\Delta^2) = -\frac{4\pi\eta F(\Delta^2)}{\Delta^2}\,, \tag{17}
$$

where $V_F(\Delta^2)$ it the Fourier transformation of the potential $V(r)$, and $F(\Delta^2)$ is the atomic form factor, which differs essentially from unity at $\Delta \lesssim 1/r_{scr}$. Thus, the Born amplitude reads

$$
M^B_{\mu_p\mu_q\lambda_1\lambda_2} = -V_F(\Delta^2)\,\mathcal{T}_{\mu_p\mu_q\lambda_1\lambda_2}\,, \tag{18}
$$

where $\mathcal{T}_{\mu_p\mu_q\lambda_1\lambda_2}$ coincides with that in Eq. (16).

If $\Delta_\perp \gg \max(r_{scr}^{-1}, |\Delta_\parallel|)$ then we can neglect the effect of screening, replace $V(r)$ by the Coulomb potential $V_c(r) = -\eta/r$, and neglect also $\Delta_\parallel = \boldsymbol{\nu}\cdot\boldsymbol{\Delta}$. Within our precision

$$
\Delta_\parallel = -\frac{1}{2}\left[q\theta_q^2 + \omega_1\theta_{k_1}^2 + \omega_2\theta_{k_2}^2 + \frac{m^2(\omega_1+\omega_2)}{pq}\right]\,. \tag{19}
$$

A simple calculation gives for the factor $A_0(\boldsymbol{\Delta})$:

$$
A_0(\boldsymbol{\Delta}) = -\frac{4\pi\eta(L\Delta)^{2i\eta}}{\Delta^2}\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)}\,, \tag{20}
$$

where $\Gamma(x)$ is the Euler Γ function and $L \sim \min(|\Delta_\parallel|^{-1}, r_{scr})$. Note that the factor $(L\Delta)^{2i\eta}$ is irrelevant because it disappears in $|M|^2$. Thus, in the region $\Delta_\perp \gg \max(r_{scr}^{-1}, |\Delta_\parallel|)$, we have $|A_0(\boldsymbol{\Delta})| = |A_0^B(\boldsymbol{\Delta})|$.

Let us represent the cross section $d\sigma$ (1) as a sum of the Born term and the Coulomb corrections:

$$
\begin{aligned}
&d\sigma_{\mu_p\mu_q\lambda_1\lambda_2} = d\sigma^B_{\mu_p\mu_q\lambda_1\lambda_2} + d\sigma^C_{\mu_p\mu_q\lambda_1\lambda_2}\,,\\
&d\sigma^B_{\mu_p\mu_q\lambda_1\lambda_2} = \frac{\alpha^2}{(2\pi)^6}\omega_1\omega_2\, d\omega_1 d\omega_2\, d\boldsymbol{\theta}_{k_1}\, d\boldsymbol{\theta}_{k_2} d\boldsymbol{\Delta}_\perp\, |A_0^B(\boldsymbol{\Delta})|^2 |\mathcal{T}_{\mu_p\mu_q\lambda_1\lambda_2}|^2\,,\\
&d\sigma^C_{\mu_p\mu_q\lambda_1\lambda_2} = \frac{\alpha^2}{(2\pi)^6}\omega_1\omega_2\, d\omega_1 d\omega_2\, d\boldsymbol{\theta}_{k_1}\, d\boldsymbol{\theta}_{k_2} d\boldsymbol{\Delta}_\perp\, R(\boldsymbol{\Delta})\, |\mathcal{T}_{\mu_p\mu_q\lambda_1\lambda_2}|^2\,,\\
&R(\boldsymbol{\Delta}) = |A_0(\boldsymbol{\Delta})|^2 - |A_0^B(\boldsymbol{\Delta})|^2\,,
\end{aligned} \tag{21}
$$

where we pass from the integration over $d\Omega_{\boldsymbol{q}}$ to the integration over $d\boldsymbol{\Delta}_\perp$. It is seen from Eqs. (17) and (20) that only the region of small $\Delta_\perp$, $\Delta_\perp \sim \max(r_{scr}^{-1}, |\Delta_\parallel|) \ll m$, gives the contribution to $d\sigma^C$. The term $\boldsymbol{A}(\boldsymbol{\Delta})$ coincides with the corresponding term in the single bremsstrahlung cross section [18]. As shown in Ref. [18], the function $R(\boldsymbol{\Delta})$ is very sensitive to the shape of the atomic potential at $r \sim r_{scr}$, while the integral,

$$
\begin{aligned}
&\int d\boldsymbol{\Delta}_\perp\, \boldsymbol{\Delta}_\perp^2 R(\boldsymbol{\Delta}) = -32\pi^3\eta^2 f(\eta)\,,\\
&f(\eta) = \mathrm{Re}\,\psi(1+i\eta) - \psi(1)\,,
\end{aligned} \tag{22}
$$

is independent of this shape; $\psi(x) = d\ln\Gamma(x)/dx$. Therefore, the Coulomb corrections integrated over $d\boldsymbol{\Delta}_\perp$ have the form

$$
d\sigma^C_{\mu_p\mu_q\lambda_1\lambda_2} = -\frac{\alpha^2\eta^2 f(\eta)}{4\pi^3}\omega_1\omega_2\, d\omega_1 d\omega_2\, d\boldsymbol{\theta}_{k_1}\, d\boldsymbol{\theta}_{k_2}\, |\boldsymbol{T}^{(0)}_{\mu_p\mu_q\lambda_1\lambda_2}(\boldsymbol{k}_1,\boldsymbol{k}_2) + \boldsymbol{T}^{(0)}_{\mu_p\mu_q\lambda_2\lambda_1}(\boldsymbol{k}_2,\boldsymbol{k}_1)|^2\,, \tag{23}
$$

where the function $\boldsymbol{T}^{(0)}_{\mu_p\mu_q\lambda_1\lambda_2}(\boldsymbol{k}_1,\boldsymbol{k}_2)$ is $\boldsymbol{T}_{\mu_p\mu_q\lambda_1\lambda_2}(\boldsymbol{k}_1,\boldsymbol{k}_2)$, Eq. (13), taken at $\boldsymbol{\Delta}_\perp = 0$, i.e., at $\boldsymbol{\theta}_q = -(\omega_1\boldsymbol{\theta}_{k_1} + \omega_2\boldsymbol{\theta}_{k_2})/q$. The main contribution to the Born cross section integrated over $d\boldsymbol{\Delta}_\perp$ is given by the region $m \gg \Delta_\perp \gg m\beta$ of small $\Delta_\perp$, where

$$
\beta = \max\left\{\frac{1}{mr_{scr}}\,,\ \frac{|\Delta_\parallel|}{m}\right\}\,. \tag{24}
$$

Assuming that $\ln(1/\beta) \gg 1$, we have within logarithmic accuracy:

$$
\begin{aligned}
d\sigma^B_{\mu_p\mu_q\lambda_1\lambda_2} &= \frac{\alpha^2\eta^2}{4\pi^3}\omega_1\omega_2\, d\omega_1 d\omega_2\, d\boldsymbol{\theta}_{k_1}\, d\boldsymbol{\theta}_{k_2}\, \ln\frac{1}{\beta}\\
&\times |\boldsymbol{T}^{(0)}_{\mu_p\mu_q\lambda_1\lambda_2}(\boldsymbol{k}_1,\boldsymbol{k}_2) + \boldsymbol{T}^{(0)}_{\mu_p\mu_q\lambda_2\lambda_1}(\boldsymbol{k}_2,\boldsymbol{k}_1)|^2\,.
\end{aligned} \tag{25}
$$

In order to demonstrate the angular dependence of the Coulomb corrections, we introduce the dimensionless quantity S,

$$
S = \frac{m^6}{2}\sum_{\mu_p\mu_q\lambda_1\lambda_2} |\boldsymbol{T}^{(0)}_{\mu_p\mu_q\lambda_1\lambda_2}(\boldsymbol{k}_1,\boldsymbol{k}_2) + \boldsymbol{T}^{(0)}_{\mu_p\mu_q\lambda_2\lambda_1}(\boldsymbol{k}_2,\boldsymbol{k}_1)|^2\,, \tag{26}
$$

and show in Fig. 1 the dependence of S on $\delta_2 = p\theta_{k_2}/m$ at fixed $\delta_1 = p\theta_{k_1}/m$, ω_1/ε_p, ω_2/ε_p, and the azimuth angle ϕ between vectors $\boldsymbol{\theta}_{k_1}$ and $\boldsymbol{\theta}_{k_2}$.

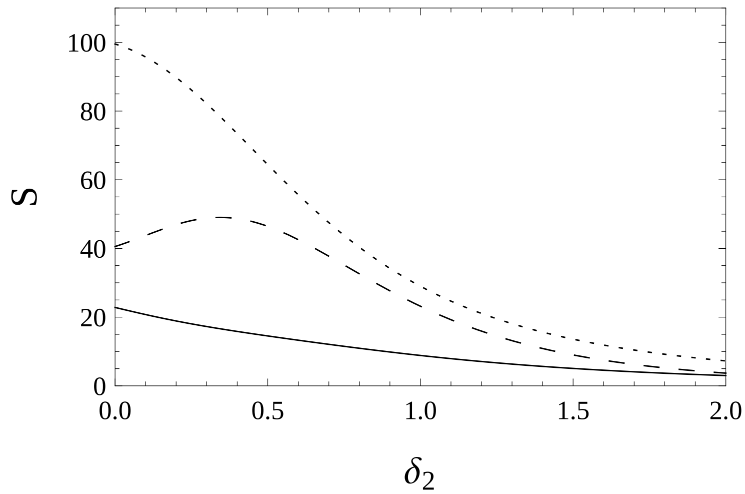

FIG. 1: The quantity S (26) as a function of $\delta_2 = p\theta_{k_2}/m$ at $\omega_1/\varepsilon_p = 0.2$, $\omega_2/\varepsilon_p = 0.4$, $\phi = 0$, $\delta_1 = p\theta_{k_1}/m = 0.2$ (dashed curve), $\delta_1 = 1$ (dotted curve), and $\delta_1 = 2$ (solid curve).

In Fig. 2 the quantity S is shown as a function of ϕ at fixed $\delta_1 = p\theta_{k_1}/m$, $\delta_2 = p\theta_{k_2}/m$, ω_1/ε_p, and ω_2/ε_p. Note that S is invariant under the replacement $\phi \to -\phi$.

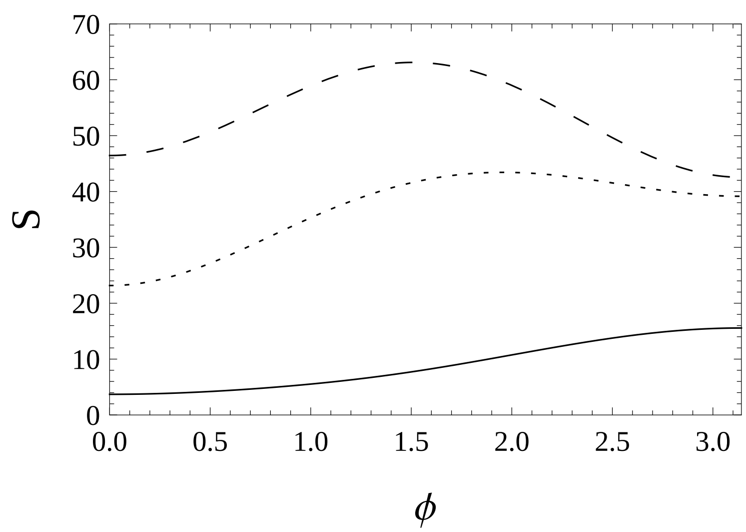

FIG. 2: The quantity S (26) as a function of the azimuth angle ϕ between vectors $\boldsymbol{\theta}_{k_1}$ and $\boldsymbol{\theta}_{k_2}$ at $\omega_1/\varepsilon_p = 0.2$, $\omega_2/\varepsilon_p = 0.4$, $\delta_1 = 0.2$, $\delta_2 = 0.5$ (dashed curve), $\delta_2 = 1$(dotted curve), and $\delta_2 = 2$ (solid curve).

It is seen from Figs. 1 and 2 that S has a smooth angular dependence. In Fig. 3 we show

the dependence of the quantity S_1 on δ_1 at fixed ω_1/ε_p and ω_2/ε_p, where

$$S_1 = \frac{p^2}{16\pi^2 m^2} \int S\, d\boldsymbol{\theta}_{k_2}\,. \tag{27}$$

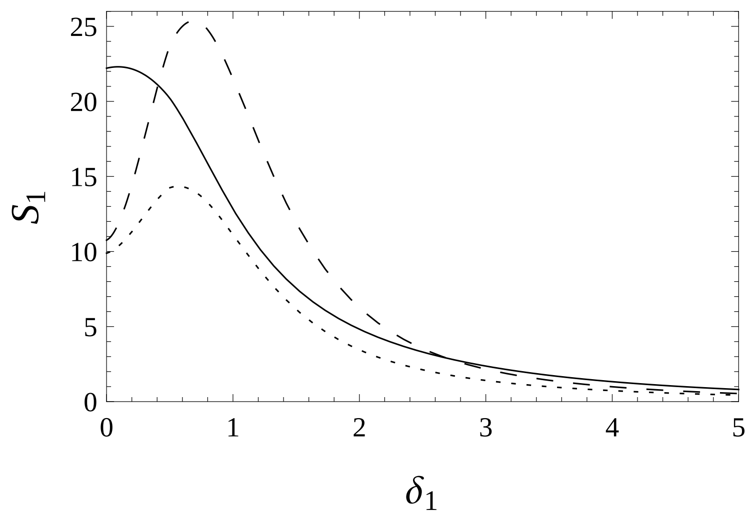

FIG. 3: The quantity S_1 (27) as a function of δ_1 at $\omega_1/\varepsilon_p = \Omega x$ and $\omega_2/\varepsilon_p = \Omega(1-x)$, where $\Omega = 0.4$, $x = 0.3$ (dashed curve), $x = 0.5$ (dotted curve), and $x = 0.7$ (solid curve).

It is seen that the main contribution to the cross section is given by the region $\delta_1 \sim 1$.

Let us discuss now the Coulomb corrections to the cross section integrated over $\boldsymbol{\theta}_{k_1}$ and $\boldsymbol{\theta}_{k_2}$ (the spectrum), averaged over the polarization of the initial electron polarization, and summed over polarizations of the final particles. We write it as

$$d\sigma^C = -\frac{8\alpha^2\eta^2 f(\eta) d\omega_1 d\omega_2}{\pi m^2 \omega_1 \omega_2} G(\omega_1/\varepsilon_p,\, \omega_2/\varepsilon_p)\,, \tag{28}$$

where the function $f(\eta)$ is qiven in Eq. (22). For $\omega_2 \ll \omega_1,\, \varepsilon_q$, a simple calculation gives the result, which corresponds to the soft-photon-emission approximation [3]:

$$\begin{aligned} F(x) = G(x,0) &= \int_0^\infty \frac{dy}{(1+y)^2}\left[1 + (1-x)^2 - \frac{4y(1-x)}{(1+y)^2}\right]\Phi(x,y)\,, \\ \Phi(x,y) &= \frac{t}{\sqrt{t^2-1}}\ln(t + \sqrt{t^2-1}) - 1\,, \quad t = 1 + \frac{x^2(1+y)}{2(1-x)}\,. \end{aligned} \tag{29}$$

The function $F(x)$ is shown in Fig. 4. The asymtotic behavior of the function $F(x)$ is

$$\begin{aligned} F(x) &\approx \frac{4}{3}x^2 \ln\frac{1}{x} \quad \text{at} \quad x \ll 1\,, \\ F(x) &\approx \ln\frac{1}{1-x} \quad \text{at} \quad 1 - x \ll 1\,. \end{aligned} \tag{30}$$

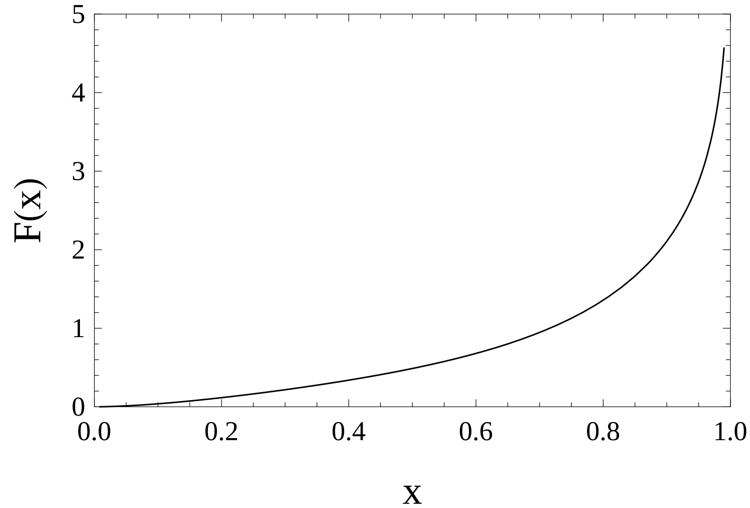

FIG. 4: Dependence of $F(x)$, Eq. (29), on $x = \omega_1/\varepsilon_p$.

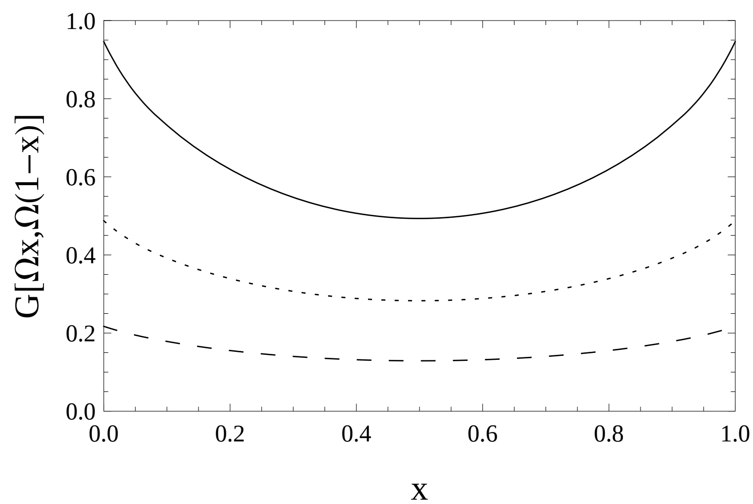

FIG. 5: Dependence of $G[\Omega x, \Omega(1-x)]$, Eq. (29), on x at $\Omega = 0.3$ (dashed curve), $\Omega = 0.5$ (dotted curve), and $\Omega = 0.7$ (solid curve). Here $\Omega = (\omega_1+\omega_2)/\varepsilon_p$ and $x = \omega_1/(\omega_1+\omega_2)$.

In Fig. 5 we show the dependence of the function $G[\Omega x, \Omega(1-x)]$ on x at fixed values of Ω, where $\Omega = (\omega_1+\omega_2)/\varepsilon_p$ and $x = \omega_1/(\omega_1+\omega_2)$.

Within logarithmic accuracy we also have for the Born cross section

$$d\sigma^B = \frac{8\alpha^2\eta^2 d\omega_1 d\omega_2}{\pi m^2 \omega_1 \omega_2}\, G(\omega_1/\varepsilon_p,\, \omega_2/\varepsilon_p)\, \ln\frac{1}{\beta_0}\,, \tag{31}$$

where the function G is the same as in Eq. (28), and

$$\beta_0 = \max\left\{\frac{1}{m r_{scr}}\,,\ \frac{m(\omega_1+\omega_2)}{\varepsilon_p\varepsilon_q}\right\} \ll 1\,. \tag{32}$$

IV. CONCLUSION

We have investigated in detail the process of high-energy double bremsstrahlung in the field of a heavy atom. The results, Eq. (13), are exact in the parameters of the atomic field and are valid even for $\eta \sim 1$. The Coulomb corrections to the differential cross section are very sensitive to the shape of the atomic potential, while the Coulomb corrections to the cross section, integrated over the momentum transfer $\mathbf{\Delta}_\perp$, are the universal function of η. It is shown that, similar to the case of single bremsstrahlung, the potential enters the amplitudes of high-energy double bremsstrahlung via the factor $\boldsymbol{A}(\mathbf{\Delta}_\perp)$. Note that such factorization takes place only for the cross section obtained in the leading quasiclassical approximation and is violated by the first quasiclassical correction. It follows from the result of Ref. [23] that the main contribution to the first quasiclassical correction to the cross section is given by the region $\Delta \sim m$. The factorized form of the amplitudes (13) and also of the amplitudes of single bremsstrahlung allows us to formulate the recipe for the calculation of the multiple bremsstrahlung differential cross section. In order to obtain the amplitude of this process exactly in the parameter η for any shape of the atomic potential $V(r)$, it is sufficient to derive the amplitude in the Born approximation and then to replace in this amplitude the Fourier transform $V_F(\Delta^2)$ of the potential $V(r)$ by the impact-factor $A_0(\mathbf{\Delta}_\perp)$ (15). Our recipe extends the impact-factor approach of Ref. [31] to the region of small momentum transfer. Note that it is just the region where the Coulomb corrections to the cross section of bremsstrahlung come from. We stress that our formulas for the cross sections of high-energy double bremsstrahlung are obtained exactly in the parameter $\eta = Z\alpha$ and, in particular, valid for $Z \gg 1$. This is important for analysis of experimental data from modern detectors, where high-Z materials are widely used.

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Appendix

In this Appendix, following the method of [18], we consider the calculation of the quantity $\mathcal{M}$,

$$\mathcal{M} = \iint d\boldsymbol{r}_1 d\boldsymbol{r}_2 \, e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}_1 - i\boldsymbol{k}_2\cdot\boldsymbol{r}_2} f_0(\boldsymbol{r}_2) d_0(\boldsymbol{r}_2, \boldsymbol{r}_1) g_0(\boldsymbol{r}_1)\,, \tag{33}$$

which contributes to the amplitude (12). Other quantities are calculated in the same way. The functions d_0, f_0, and g_0 are given in Eq. (6), Eq. (7), and Eq. (8), respectively.

We split the integration region into three, $z_1 < z_2 < 0$, $z_1 < 0 \,\&\, z_2 > 0$, $z_2 > z_1 > 0$, and denote the corresponding contributions to $\mathcal{M}$ as $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$. In the first region, the functions g_0 and d_0 have simple eikonal forms

$$\begin{aligned}
& g_0(\boldsymbol{r}_1) = e^{i\boldsymbol{p}\cdot\boldsymbol{r}_1} \exp\left[-i\int_0^\infty dx V(\boldsymbol{r}_1 - x\boldsymbol{n}_{\boldsymbol{p}})\right]\,, \\
& d_0(\boldsymbol{r}_2, \boldsymbol{r}_1) = -\frac{e^{i\kappa r}}{4\pi r} \exp\left[-ir\int_0^1 dx V(\boldsymbol{r}_1 + x\boldsymbol{r})\right]\,, \\
& \boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1\,, \quad \kappa = \sqrt{(\varepsilon_p - \omega_1)^2 - m^2}\,,
\end{aligned} \tag{34}$$

so that

$$\begin{aligned}
& \mathcal{M}_1 = \frac{i}{(2\pi)^2} \iint\limits_{z_1<z_2<0} \frac{d\boldsymbol{r}_1 d\boldsymbol{r}_2}{r} \int d\boldsymbol{Q}\, \exp(i\Phi)\,, \\
& \Phi = Q^2 + (\boldsymbol{p} - \boldsymbol{k}_1)\cdot\boldsymbol{r}_1 - (\boldsymbol{q} + \boldsymbol{k}_2)\cdot\boldsymbol{r}_2 + \kappa r \\
& - \int_0^\infty dx\, V(\boldsymbol{r}_1 - x\boldsymbol{n}_p) - r\int_0^1 dx V(\boldsymbol{r}_1 + x\boldsymbol{r}) - \int_0^\infty dx\, V(\boldsymbol{r}_q)\,, \\
& \boldsymbol{r}_q = \boldsymbol{r}_2 + x\boldsymbol{n}_q + \boldsymbol{Q}\sqrt{\frac{2|\boldsymbol{n}_q\cdot\boldsymbol{r}_2|}{q}}\,.
\end{aligned} \tag{35}$$

Within our accuracy we can replace the quantity $V(\boldsymbol{r}_1 - x\boldsymbol{n}_p)$ and $V(\boldsymbol{r}_1 + x\boldsymbol{r})$ in (35) by $V(\boldsymbol{r}_1 - x\boldsymbol{n}_p + \boldsymbol{Q}\sqrt{2|\boldsymbol{n}_q\cdot\boldsymbol{r}_2|/q})$ and $V(\boldsymbol{r}_1 + x\boldsymbol{r} + \boldsymbol{Q}\sqrt{2|\boldsymbol{n}_q\cdot\boldsymbol{r}_2|/q})$, respectively, shift $\boldsymbol{\rho}_1 \to \boldsymbol{\rho}_1 - \boldsymbol{Q}\sqrt{2|\boldsymbol{n}_q\cdot\boldsymbol{r}_2|/q}$, $\boldsymbol{\rho}_2 \to \boldsymbol{\rho}_2 - \boldsymbol{Q}\sqrt{2|\boldsymbol{n}_q\cdot\boldsymbol{r}_2|/q}$, where $\boldsymbol{\rho}_1 = \boldsymbol{r}_{1\perp}$ and $\boldsymbol{\rho}_2 = \boldsymbol{r}_{2\perp}$. Then we take the integral over $\boldsymbol{Q}$ and obtain

$$\begin{aligned}
& \mathcal{M}_1 = -\frac{1}{4\pi} \iint\limits_{z_1<z_2<0} \frac{d\boldsymbol{r}_1 d\boldsymbol{r}_2}{r} \exp[i(\Phi_0 + \Phi_1)]\,, \\
& \Phi_0 = (\boldsymbol{p} - \boldsymbol{k}_1)\cdot\boldsymbol{r}_1 - (\boldsymbol{q} + \boldsymbol{k}_2)\cdot\boldsymbol{r}_2 + \kappa r
\end{aligned}$$

$$
-\int_0^\infty dx\, V(\boldsymbol{r}_1 - x\boldsymbol{n}_p) - r\int_0^1 dx V(\boldsymbol{r}_1 + x\boldsymbol{r}) - \int_0^\infty dx\, V(\boldsymbol{r}_2 + x\boldsymbol{n}_q)\,,
$$

$$
\Phi_1 = -\frac{\Delta_\perp^2 |\boldsymbol{n}_q \cdot \boldsymbol{r}_2|}{2q}\,. \tag{36}
$$

In the same way, we obtain for $\mathcal{M}_2$ and $\mathcal{M}_3$,

$$
\begin{aligned}
\mathcal{M}_2 &= -\frac{1}{4\pi}\iint\limits_{z_2>0, z_1<0} \frac{d\boldsymbol{r}_1 d\boldsymbol{r}_2}{r}\,\exp[i(\Phi_0 + \Phi_2)]\,,\\
\Phi_2 &= -\frac{\Delta_\perp^2 |\boldsymbol{r}\cdot\boldsymbol{r}_1||\boldsymbol{r}\cdot\boldsymbol{r}_2|}{2\kappa r^3}\,,\\
\mathcal{M}_3 &= -\frac{1}{4\pi}\iint\limits_{z_2>z_1>0} \frac{d\boldsymbol{r}_1 d\boldsymbol{r}_2}{r}\,\exp[i(\Phi_0 + \Phi_3)]\,,\\
\Phi_3 &= -\frac{\Delta_\perp^2 |\boldsymbol{n}_p\cdot\boldsymbol{r}_1|}{2p}\,.
\end{aligned} \tag{37}
$$

There are two overlapping regions of the momentum transfer Δ:

$$
\begin{aligned}
&\mathrm{I.}\, \Delta \gg \frac{m^2(\omega_1+\omega_2)}{\varepsilon_p\varepsilon_q}\\
&\mathrm{II.}\, \Delta \ll \frac{m(\omega_1+\omega_2)}{\varepsilon_p}\,.
\end{aligned} \tag{38}
$$

In the first region we can neglect in the phase Φ_0 the term $\Delta_\parallel$ as compared with $\Delta_\perp$ and replace in the integrals $\boldsymbol{n}_q \to \boldsymbol{\nu}$ and $\boldsymbol{r} \to (\boldsymbol{\nu}\cdot\boldsymbol{r})\boldsymbol{\nu}$, where z axes is parallel to $\boldsymbol{\nu} = \boldsymbol{n}_p$. Performing the integration over z_1, z_2, and $\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$, we obtain

$$
\mathcal{M} = \frac{i}{2}\int d\boldsymbol{\rho}\exp\left[-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{\rho} - i\chi(\rho)\right]\left[qN_1 - \kappa N_2 - pN_3\right], \tag{39}
$$

where the quantities $\chi(\rho)$, N_1, N_2, and N_3 are defined in Eq. (14). Then we use the relation

$$
qN_1 - \kappa N_2 - pN_3 = \boldsymbol{\Delta}_\perp\cdot\boldsymbol{j}_0\,, \tag{40}
$$

where $\boldsymbol{j}_0$ is given in Eq. (14). Performing integration by parts, we finally obtain $\mathcal{M}$ in the first region:

$$
\begin{aligned}
\mathcal{M} &= -\frac{i}{2}\int d\boldsymbol{\rho}\exp\left[-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{\rho} - i\chi(\rho)\right]\boldsymbol{\nabla}_\perp\chi(\rho)\cdot\boldsymbol{j}_0\\
&= -\frac{i}{2}\int d\boldsymbol{r}\exp\left[-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{\rho} - i\chi(\rho)\right]\boldsymbol{\nabla}_\perp V(r)\cdot\boldsymbol{j}_0\,.
\end{aligned} \tag{41}
$$

In the second region, one can neglect the term Φ_1 in Eq. (36) and the terms $\Phi_{2,3}$ in Eq. (37). In the phase Φ_0 we take into account the linear terms of expansion of the integrals in $\boldsymbol{n}_q - \boldsymbol{\nu}$

and $\boldsymbol{r}-(\boldsymbol{r}\cdot\boldsymbol{\nu})\boldsymbol{\nu}$. The result, which is valid both in the region I and in the region II, has the form

$$\mathcal{M}=-\frac{i}{2}\int d\boldsymbol{r}\exp\left[-i\boldsymbol{\Delta}\cdot\boldsymbol{r}-i\chi(\rho)\right]\boldsymbol{\nabla}_{\perp}V(r)\cdot\boldsymbol{j}_0\,. \tag{42}$$

It corresponds to the second line in Eq. (41) with the replacement $\boldsymbol{\Delta}_{\perp}\cdot\boldsymbol{\rho}\to\boldsymbol{\Delta}\cdot\boldsymbol{r}$.

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